2. Preliminaries

2.1. Syntax of Equations

First, we define how to construct terms and equations.

Def. 2.1.1. (Signature)
A signature $\Sigma = \bigcup_{i \in \mathbb{N}} \Sigma_i$ is a union of pairwise disjoint finite sets Σ_i. Every $f \in \Sigma_i$ is called a function symbol of arity i.
We always require that Σ is finite and $\Sigma_0 \neq \emptyset$.

Ex. 2.1.2 An example for a signature is
$\Sigma_0 = \{0\}$, $\Sigma_1 = \{succ\}$, $\Sigma_2 = \{plus, times\}$

Def 2.1.3. (Terms) Let Σ be a signature and let $\mathcal{V}$ be a non-empty infinite set of variables with $\Sigma \cap \mathcal{V} = \emptyset$.
Then $\mathcal{T}(\Sigma, \mathcal{V})$ is the set of all terms (over Σ and $\mathcal{V}$).
It is the smallest set such that:

- $\mathcal{V} \subseteq \mathcal{T}(\Sigma, \mathcal{V})$

- $f(t_1, \ldots, t_n) \in \mathcal{T}(\Sigma, \mathcal{V})$, if $f \in \Sigma_n$ and all $t_i \in \mathcal{T}(\Sigma, \mathcal{V})$

$\mathcal{T}(\Sigma)$ stands for $\mathcal{T}(\Sigma, \varnothing)$, i.e., for the set of ground terms.

Ex 2.14 $\left.\begin{array}{l} 0,\ succ(0),\ succ(succ(0)), \\ plus(0, succ(0)), \ldots \end{array}\right\} \in \mathcal{T}(\Sigma)$

$plus(times(0, x), succ(y)) \in \mathcal{T}(\Sigma, \mathcal{V})$

Def 2.13 (cont.) For a term t, $\mathcal{V}(t)$ is the set of all variables in t. For $T \subseteq \mathcal{T}(\Sigma, \mathcal{V})$, $\mathcal{V}(T) = \bigcup_{t \in T} \mathcal{V}(t)$.

A term q is a subterm of a term t (denoted $t \trianglerighteq q$) iff $t = q$ or $t = f(t_1, \ldots, t_n)$ and $t_i \trianglerighteq q$ for some $i \in \{1, \ldots, n\}$.

↑ "if and only if"

A term q is a proper subterm of t (denoted $t \triangleright q$) iff $t \trianglerighteq q$ and $t \neq q$.

Ex. 2.14 (cont)

$t = plus(times(0, x), succ(y))$

$\mathcal{V}(t) = \{x, y\}$

$t \triangleright times(\Theta, X)$

$t \triangleright y$ etc.

Def 2.15. (Equations)

For $t_1, t_2 \in \mathcal{T}(\Sigma, \mathcal{V})$, an *equation* is an expression of the form $t_1 \equiv t_2$.

So "$\equiv$" is used as a syntactic equality symbol. It does not mean that "$t_1 \equiv t_2$" is true.

So an equation is just a pair of terms, but we usually write "$t_1 \equiv t_2$" instead of (t_1, t_2)

Thus, a *set of equations* is a subset of $\mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$